Normal families

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<u>Theorem</u> (Arzela-Ascoli) (Precompactness criterium)

Let $\mathcal{F}$ be a family of continuous functions $f: \Omega \to \mathbb{C}$ (Ω - a region)

Then any sequence of functions (f_n) from $\mathcal{F}$ contains locally uniformly convergent subsequence (f_{n_k}) if and only if

1) $\mathcal{F}$ is <u>uniformly bounded on compacts</u>: $\forall K \subset \Omega$ - compact

and $\exists M > 0 : \forall z \in K \ \forall f \in \mathcal{F} \ |f(z)| \le M$.

2) $\mathcal{F}$ is <u>uniformly equicontinuous on compacts</u>: $\forall K \subset \Omega$ - compact

$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall z_1, z_2 \in K, \ \forall f \in \mathcal{F} : |z_1 - z_2| < \delta \Rightarrow |f(z_1) - f(z_2)| < \varepsilon$.

<u>Proof</u> ($\Downarrow$) If $\mathcal{F}$ is <u>not uniformly bounded</u> on some compact K, then $\exists z_n \in K$, $f_n \in \mathcal{F}$: $|f_n(z_n)| \ge n$.

If (f_{n_k}) - uniformly convergent on K subsequence of (f_n),

$f_{n_k} \rightrightarrows f$, then $\exists k: \kappa \ge k \Rightarrow |f_{n_\kappa}(z) - f(z)| < 1 \ \forall z \in K \Rightarrow$

$|f_{n_\kappa}(z)|$ is bounded (by $\max\limits_{z \in K} |f(z)| + 1$)

Take $n_\kappa > \max\limits_{z \in K} |f(z)| + 1$ to arrive to contradiction

If $\mathcal{F}$ is <u>not uniformly equicontinuous</u> then

$\exists \varepsilon > 0 \ \forall n \in \mathbb{N} \ \exists f_n \in \mathcal{F}, \ z_n, w_n \in K: |z_n - w_n| < \frac{1}{n}, \ |f_n(z_n) - f_n(w_n)| \ge \varepsilon$.

Let $f_{n_k} \rightrightarrows f$, f is uniformly continuous, so $\exists \delta > 0 : |z - w| < \delta \Rightarrow |f(z) - f(w)| < \frac{\varepsilon}{3}$

Also $\exists k : \kappa > k \Rightarrow |f_{n_\kappa}(z) - f(z)| < \frac{\varepsilon}{3}$. So if we pick $\frac{1}{n_\kappa} < \delta$, we get

$|f_{n_\kappa}(z_{n_\kappa}) - f_{n_\kappa}(w_{n_\kappa})| \le |f_{n_\kappa}(z_{n_\kappa}) - f(z_{n_\kappa})| + |f(z_{n_\kappa}) - f(w_{n_\kappa})| + |f(w_{n_\kappa}) - f_{n_\kappa}(w_{n_\kappa})| < \varepsilon$.

Contradiction!

($\Uparrow$) Let $(\xi_k) \subset \Omega$ be a dense sequence of points (i.e. $\operatorname{Clos}\{\xi_k\} = \operatorname{Clos}(\Omega)$).

Let $(f_n) \subset \mathcal{F}$ be a sequence.

Since $(f_n(\xi_1))$ is a bounded sequence, it has a convergent subsequence $(f_{n,1}(\xi_1))$

Since $(f_{n,1}(\xi_2))$ is a bounded sequence, it has a convergent subsequence $(f_{n,2}(\xi_2))$. Also, $(f_{n,2}(\xi_1))$ converge (as a subsequence of $(f_{n,1}(\xi_1))$).

...

Repeat to get $(f_{n,m})_{n=1}^{\infty}$, such that $\forall j \le m$, $(f_{n,m}(\xi_j))_{n=1}^{\infty}$ is a convergent sequence.

Define $g_n := f_{n,n}$. Then $\forall i$: $(g_n(\xi_i))$ is convergent

$(f_{n,m}(\zeta_i))_{m=1}^{\infty}$ is a convergent sequence.

Define $g_k := f_{k,k}$. Then $\forall i$ $(g_k(\zeta_i))$ is convergent,
since $\forall j$ for $k \geq i$, $(g_k(\zeta_i)) = (f_{k,k}(\zeta_i))$ is a subsequence
of convergent $(f_{n,i}(\zeta_i))_{n=1}^{\infty}$.

Let us prove that g_k converges locally uniformly in Ω.
By the definition of local uniform convergence

Local Uniform Convergence

we only need to prove that $\forall z \in \Omega$ $\forall \varepsilon > 0$ $\exists \delta(\varepsilon, z) > 0, N(\varepsilon, z)$:
$n, m > N(\varepsilon, z) \Rightarrow \forall w \in B(z, \delta)$ $|g_n(w) - g_m(w)| < \varepsilon$.

Bonus (+1 pt). Mistake in Ahlfors in this proof

Fix $\varepsilon > 0$, $z \in \Omega$. Let $r < dist(z, \partial\Omega)$
$\mathcal{F}$ is equicontinuos on compact $\overline{B(z,r)} \subset \Omega$, so
$\exists r > \delta > 0$: $|w_1 - w_2| < 2\delta$, $w_1, w_2 \in B(z,r) \Rightarrow \forall f \in \mathcal{F}$, $|f(w_1) - f(w_2)| < \frac{\varepsilon}{3}$.
Consider $B(z, \delta)$. $\exists k$: $\zeta_k \in B(z, \delta)$ — dense!
$\exists N$: $n, m > N$ $|g_n(\zeta_k) - g_m(\zeta_k)| < \frac{\varepsilon}{3}$.
Then $\forall w \in B(z, \delta)$: $|g_n(w) - g_m(w)| \leq \underbrace{|g_n(w) - g_n(\zeta)|}_{< \frac{\varepsilon}{3} \text{ (since } |w-\zeta| < 2\delta)} + \underbrace{|g_n(\zeta) - g_m(\zeta)|}_{< \frac{\varepsilon}{3} \ (n,m > N)} + \underbrace{|g_m(\zeta) - g_m(w)|}_{< \frac{\varepsilon}{3} \ (|w-\zeta| < 2\delta)} < \varepsilon$.

Paul Montel

Def Let Ω be a region, $\mathcal{F} \subset A(\Omega)$ — a family of analytic functions is called normal if $\forall$ sequence $(f_n) \subset \mathcal{F}$ $\exists$ a subsequence (f_{n_k}) converging locally uniformly.

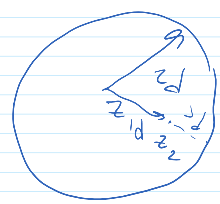
Theorem (Montel) $\mathcal{F}$ is normal iff

Theorem (Montel) $\mathcal{F}$ is normal iff it is uniformly bounded on compacts.

Proof. If $\mathcal{F}$ is not uniformly bounded on some $K \subset \Omega$ - compact then $\exists (f_n) \subset \mathcal{F}$, $z_n \in \Omega$: $|f_n(z_n)| \to \infty$ In particular, for any subsequence $|f_{n_k}(z_{n_k})| \to \infty$, so for any $f \in \mathcal{A}(\Omega)$, $\sup_{z \in K} |f_{n_k}(z) - f(z)| \geq |f_{n_k}(z_{n_k}) - f(z_{n_k})| \to \infty$ - does not converge!

Let $\mathcal{F}$ be uniformly bounded on compacts. By Arzela Theorem, we need to prove equicontinuity on compacts.
Let $K \subset \Omega$ - compact. $z \to \operatorname{dist}(z, \partial\Omega)$ - continuous on K, so it reaches minimum.
So $\exists\, d > 0$: $\forall z \in K$ $\operatorname{dist}(z, \partial\Omega) > 4d \Rightarrow \overline{B(z, 2d)} \subset \Omega$.
Let $F := \{z \in \mathbb{C} : \operatorname{dist}(z, K) \leq 2d\} \subset \Omega$. closed, bounded, so F is compact.
Let $M := \max\{|f(z)| : z \in F\}$.

If $z_1, z_2 \in K$, $|z_1 - z_2| < d$, consider $C_{2d} = \{z : |z - z_1| = 2d\}$, positively oriented. Then $n(C_{2d}, z_1) = n(C_{2d}, z_2) = 1$. $C_{2d} \subset F$.



$$f(z_1) = \frac{1}{2\pi i} \oint_{C_{2d}} \frac{f(\zeta)}{\zeta - z_1}\, d\zeta \qquad f(z_2) = \frac{1}{2\pi i} \oint_{C_{2d}} \frac{f(\zeta)}{\zeta - z_2}\, d\zeta$$

$$f(z_1) - f(z_2) = \frac{z_1 - z_2}{2\pi i} \oint_{C_{2d}} \frac{f(\zeta)}{(\zeta - z_1)(\zeta - z_2)}\, d\zeta$$

So $|f(z_1) - f(z_2)| \leq \frac{1}{2\pi} |z_1 - z_2| \operatorname{length}(C_{2d}) \cdot \frac{M}{2d \cdot d} = |z_1 - z_2| \frac{M}{d}$.
(since $|\zeta - z_1| = 2d$, $|\zeta - z_1| \geq |\zeta - z_2| - |z_1 - z_2| \geq d$)

So for $\varepsilon > 0$, if $\delta = \min\left(d, \frac{\varepsilon d}{M}\right)$, then
$|z_1 - z_2| < \delta \Rightarrow |z_1 - z_2| < d$ so $|f(z_1) - f(z_2)| < \delta \frac{M}{d} = \varepsilon$. ∎

Corollary (Montel's convergence criterium).
Assume $(f_n) \subset \mathcal{A}(\Omega)$ is locally uniformly bounded. If every convergent subsequence (f_{n_k}) of (f_n) converges locally uniformly ~~to~~ f, then $f_n \to f$ locally uniformly.

Let f_n does not converge to f locally. It means $\exists K \subset \Omega$ - compact, $\varepsilon > 0$:

$\forall N\ \exists n>N: \sup_{z\in K}|f_n(z)-f(z)|\geq\varepsilon$. Take n_1: $\sup_{z\in K}|f_{n_1}(z)-f(z)|\geq\varepsilon$.

Take $n_2>n_1$: $\sup_{z\in K}|f_{n_2}(z)-f(z)|>\varepsilon$

Construct recursively $n_k>n_{k-1}$: $\sup_{z\in K}|f_{n_k}(z)-f(z)|>\varepsilon$.

Then $g_k:=f_{n_k}$ is locally uniformly bounded.

So it has a subsequence (g_{k_ℓ}) which converges on K to g.

But $\sup_{z\in K}|g(z)-f(z)|=\lim_{\ell\to\infty}|g_{k_\ell}(z)-f(z)|\geq\varepsilon$, so $g\neq f$.

But g_{k_ℓ} – subsequence of (g_k), which is a subsequence of (f_n)

So (g_{k_ℓ}) – convergent subsequence of (f_n) which does not converge to f – contradiction! ∎

Giuseppe Vitali

Theorem (Vitali)

Let Ω be a region, $(f_n)\in A(\Omega)$ be a locally uniformly bounded sequence.

TFAE:

1) (f_n) converges locally uniformly on Ω.

2) $\exists z_0\in\Omega$; $\forall k\in\mathbb{N}$ the sequence $(f_n^{(k)}(z_0))$ converges for every k.

3) The set $A=\{z\in\Omega: \lim_{n\to\infty}f_n(z)\text{ exists}\}$ has a limit point in Ω.

Proof. 1) ⇒ 2) (f_n) converges locally uniformly $\overset{\text{Weierstrass}}{\Longrightarrow}$ $(f_n^{(k)})$ converges locally uniformly ⇒ $\forall z_0\in\Omega$: $f_n^{(k)}(z_0)$ converges locally uniformly. ∎
$\forall k\in\mathbb{N}$

2) ⇒ 3) Let $r<\operatorname{dist}(z_0,\partial\Omega)\iff\overline{B(z_0,r)}\subset\Omega$.

(f_n) is bounded on $B(z_0,r)$; so $\exists M: |f_n(z)|\leq M\ \forall z\in B(z_0,r)$.
(bounded on compact $\overline{B(z_0,r)}$)

$f_n(z)=\sum a_{n,k}(z-z_0)^k$, $a_{n,k}=\dfrac{f_n^{(k)}(z_0)}{k!}$, so $|a_{n,k}|\leq Mr^{-k}$, by Cauchy inequality.

Let $a_k:=\lim_{n\to\infty}\dfrac{f_n^{(k)}(z_0)}{k!}=\lim_{k\to\infty}a_{n,k}$. $f(z):=\sum a_k z^k$.

Observe $|a_k| = \lim_{k\to\infty} |a_{n,k}| \le M r^{-k}$.

Fix $\rho < r$. Let us show that $|z-z_0| \le \rho \Rightarrow f_n(z) \to f(z)$ $\forall z \in B(z_0,\rho)$

So $B(z_0,\rho) \subset \{z : \lim_{n\to\infty} f_n(z) \text{ exists}\}$ - has a limit point.

To prove it, write

$$|f(z) - f_n(z)| \le \sum_{k=0}^{\infty} |a_{n,k} - a_k| \, |z-z_0|^k = \underbrace{\sum_{k=0}^{m} |a_{n,k} - a_k| \, |z-z_0|^k}_{\mathrm{I}} + \underbrace{\sum_{k=m+1}^{\infty} |a_{n,k} - a_k| \, |z-z_0|^k}_{\mathrm{II}}$$

Note that $\mathrm{II} \le \sum_{k=m+1}^{\infty} 2\rho^k r^{-k} M = 2M \sum_{k=m+1}^{\infty} \left(\frac{\rho}{r}\right)^k \xrightarrow[\text{as } m\to\infty]{} 0$

since $|a_{n,k} - a_n| \le |a_{n,k}| + |a_n| \le 2r^{-k}M$

Now fix $\varepsilon > 0$ and choose m so that $2M \sum_{k=m+1}^{\infty} \left(\frac{\rho}{r}\right)^k < \varepsilon/2$

Now fix N: $\forall k \le m$, $|a_{n,k} - a_k| < \frac{\varepsilon}{2(m+1)\rho^k}$ if $n \ge N$.

Then for $n \ge N$:

$$|f(z) - f_n(z)| \le \underbrace{(m+1)\frac{\varepsilon}{2(m+1)\rho^k}}_{\mathrm{I}} + \underbrace{2M \sum_{k=m+1}^{\infty} \left(\frac{\rho}{r}\right)^k}_{\mathrm{II}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \blacksquare$$

3) $\Rightarrow$ 1). Let (g_k), (h_k) be two convergent subsequences of (f_n).

Let $g := \lim g_k$, $h := \lim h_k$.

Then $\forall z \in A = \{z \in \Omega : \exists \lim_{n\to\infty} f_n(z)\}$,

$$g(z) = \lim_{k\to\infty} g_k(z) = \lim_{n\to\infty} f_n(z) = \lim_{k\to\infty} h_k(z) = h(z).$$

So, by uniqueness Thm, $g \equiv h$ in Ω.

So, by Montel's convergence criterium, $\exists \lim_{n\to\infty} f_n(z)$ $\blacksquare$